Lecture 14
Graph Theory and Circuit Analysis

- Basic Concepts of Graph Theory
- Cut-set
- Incidence Matrix
- Circuit Matrix
- Cut-set Matrix
Definition: In a connected graph $G$ of $n$ nodes (vertices), the subgraph $T$ that satisfies the following properties is called a tree.

- $T$ is connected
- $T$ contains all the vertices of $G$
- $T$ contains no circuit,
- $T$ contains exactly $n-1$ number of edges.

In every connected graph $G$ there exists at least one tree.
Let G have p separated parts $G_1, G_2, ..., G_p$, that is $G = G_1 \cup G_2 \cup ... \cup G_p$, and let $T_i$ be a tree in $G_i$, $i=1,2,...,p$, then, $T = T_1 \cup T_2 ... \cup T_p$ is called a forest of G.

**DEFINITION:** The complement of a tree is called a co-tree and the complement of a forest is called a co-forest. The edges of a tree or a forest are called branches and the edges of a co-tree or co-forest are called links (chords).
Tree & Co-tree Examples

9 possible trees and corresponding co-trees:

\[
\begin{align*}
T_1 &= \{e_2, e_3, e_4, e_5\} & T_4 &= \{e_1, e_2, e_5, e_6\} & T_7 &= \{e_2, e_3, e_5, e_6\} \\
T_1' &= \{e_1, e_6\} & T_4' &= \{e_3, e_4\} & T_7' &= \{e_1, e_4\} \\
T_2 &= \{e_1, e_2, e_4, e_6\} & T_5 &= \{e_1, e_3, e_4, e_6\} & T_8 &= \{e_1, e_2, e_4, e_5\} \\
T_2' &= \{e_3, e_5\} & T_5' &= \{e_2, e_5\} & T_8' &= \{e_3, e_6\} \\
T_3 &= \{e_1, e_3, e_5, e_6\} & T_6 &= \{e_2, e_3, e_4, e_6\} & T_9 &= \{e_1, e_3, e_4, e_5\} \\
T_4' &= \{e_2, e_4\} & T_6' &= \{e_1, e_5\} & T_9' &= \{e_2, e_6\}
\end{align*}
\]
**DEFINITION**: Let $G$ be a graph and let $b$ and $l$ be respectively the number of branches and chords of $G$, then $b$ and $l$ are called respectively the rank and the nullity of the graph.

**THEOREM**: Let $G$ have $n$ nodes, $e$ edges and $p$ connected parts, then its rank and nullity are given respectively by

\[
b = n - p
\]

and

\[
l = e - n + p
\]
**DEFINITION**: Let $G$ be a connected graph and let $T$ and $T'$ be tree and co-tree respectively, that is $G = T \cup T'$. Let a link $e' \subseteq T'$ and its unique tree path (a path which is formed by the branches of $T$) define a circuit. This circuit is called the **fundamental circuit (f-circuit)** of $G$. All such circuits defined by all the chords of $T'$ are called the fundamental circuits (f-circuits) of $G$. If $G$ is not connected, then the f-circuits are defined with respect to a forest.
f-circuit Example

- Note that the number of f-circuits is given by the nullity of $G$ and that, with respect to a chosen tree $T$ of $G$, each f-circuit contains one and only link.

Consider the following graph

f-circuits:

$c_{f1} = \{e_3, e_1, e_2\}$,
$c_{f2} = \{e_6, e_8, e_4, e_5\}$,
$c_{f3} = \{e_7, e_8, e_4, e_5\}$

Nullity of $G$

$l = e - n + p = 8 - 6 + 1 = 3$
**DEFINITION**: The cut-set of a graph $G$ is the subgraph $G_x$ of $G$ consisting of the set of edges satisfying the following properties:

- The removal of $G_x$ from $G$ reduces the rank of $G$ exactly by one.
- No proper subgraph of $G_x$ has this property.

If $G$ is connected, then the first property in the above definition can be replaced by the following phrase.

- The removal of $G_x$ from $G$ separates the given connected graph $G$ into exactly two connected subgraphs.
Consider the following graph and the following set of edges

- $G_1 = \{e_1, e_2\}$ is also a cut-set
- $G_2 = \{e_4, e_6, e_7\}$ is a cut-set
- $G_3 = \{e_2, e_3, e_4, e_8\}$ is not a cut-set, because the removal of $G_3$ from $G$ results in three connected subgraphs
- $G_4 = \{e_2, e_3, e_6\}$ is not a cut-set, because a subset of $G_4$ is cut-set
**DEFINITION**: Let G be a connected graph and let T be its tree. The branch $e_t \subseteq T$ defines a unique cut-set (a cut-set which is formed by $e_t$ and the links of G). This cut-set is called the fundamental cut-set (f-cutset) of G. All such cut-sets defined by all the branches of T are called the fundamental cut-sets (f-cutsets) of G. If G is not connected then the f-cut sets are defined with respect to a forest.

- Note that the number of fundamental cut-sets is given by the rank of G and with respect to a chosen tree T of G, each fundamental cut-set contains one and only one branch.
f-cutset example

Consider the following graph with $T=\{e_1,e_2,e_4,e_5,e_8\}$

f-cutsets:

$x_{f1}=\{e_1,e_3\} \quad x_{f2}=\{e_2,e_3\}$

$x_{f3}=\{e_4,e_6,e_7\} \quad x_{f4}=\{e_5,e_6,e_7\}$

$x_{f5}=\{e_8,e_6,e_7\}$
• The edge $e_1$ which has a direction from node $v_1$ to node $v_2$ simply indicates that any transmission from $v_1$ to $v_2$ along $e_1$ is assumed to be positive.

• Any transmission from $v_2$ to $v_1$ along $e_1$ is assumed to be negative.
**DEFINITION**: Let $e$ and $n$ represent respectively the number of edges and nodes of a graph $G$. The incidence matrix

$$A_a = [a_{ij}]_{n \times e}$$

having $n$ rows and $e$ columns with its elements are defined as

$$a_{ij} = \begin{cases} 
1 & \text{if edge } j \text{ incident to node } i \text{ and oriented "outward"} \\
-1 & \text{if edge } j \text{ incident to node } i \text{ and oriented "inward"} \\
0 & \text{if edge } j \text{ not incident to node } i 
\end{cases}$$
Incidence Matrix:

Property:

Any column of \( A \) contains exactly two nonzero entries of opposite sign.
Reduced Incidence Matrix

- **DEFINITION**: For a connected graph $G$, the matrix $A$, obtained by deleting any one of the rows of the incidence matrix $A_a$ is called the *reduced incidence matrix*.

- Note that since any column of $A_a$ contains exactly two nonzero entries of opposite sign, one can uniquely determine the incident matrix when the reduced incident matrix is given.

- Note also that the rank of $A_a$ is $n-1$. 
In a graph $G$, let $k$ be the number of circuits and let an arbitrary circuit orientation be assigned to each one of these circuits.

**DEFINITION:** The circuit matrix

$$B = \begin{bmatrix} b_{ij} \end{bmatrix}_{k \times e}$$

for a graph $G$ of $e$ edges and $k$ circuits is defined as

$$b_{ij} = \begin{cases} 
1 & \text{if edge } j \text{ incident to circuit } i \text{ with "same" orientation} \\
-1 & \text{if edge } j \text{ incident to circuit } i \text{ with "opposite" orientation} \\
0 & \text{if edge } j \text{ not incident to circuit } i 
\end{cases}$$
Consider the following graph

\[
\begin{bmatrix}
-1 & 1 & 0 & 0 & 0 & 0 \\
0 & -1 & 1 & 1 & 0 & 0 \\
-1 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 & 1 & 1 \\
-1 & 0 & 1 & 0 & 1 & 1 \\
0 & -1 & 1 & 0 & 1 & 1 \\
\end{bmatrix}
\]
\textbf{f-Circuit Matrix}

- Let $b_i$ represent the row of $B$ that corresponds to circuit $c_i$. The circuits $c_i, \ldots, c_j$ are independent if the rows $b_i, \ldots b_j$ are independent.

- **DEFINITION:** The \textit{f-circuit matrix} $B_f$ of a graph $G$ with respect to some tree $T$ is defined as the circuit matrix consisting of the fundamental circuits of $G$ only whose orientations are chosen in the same direction as that of defining links.

- The fundamental circuit matrix $B_f$ of a graph $G$ with respect to some tree $T$ can always be written as

$$B_f = [U \ B_{f12}]_{l \times e}$$
Consider the following graph with $T' = \{e_1, e_3, e_5\}$

\[
\mathbf{B}_f = \begin{bmatrix}
1 & 0 & 0 & -1 & 0 & 0 \\
0 & 1 & 0 & -1 & 1 & 0 \\
0 & 0 & 1 & 0 & -1 & 1 \\
\end{bmatrix}
\]

= $[U \quad \mathbf{B}_{f12}]$
THEOREM: If the column orderings of the circuit and incident matrices are identical then

\[ A_a B_f^T = 0 \]

\[ B_f A_a^T = 0 \]

Also

\[ A_a B^T = 0 \]

\[ B A_a^T = 0 \]
Matrices of Oriented Graphs

Consider the following graph

\[
\begin{bmatrix}
1 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & -1 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & -1 & 0 & 0 & 1 & 1 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 1 & 0 & 1 & 0 & 0 \\
0 & -1 & 1 & 0 & 1 & 0 \\
0 & 0 & -1 & 0 & 0 & 1 \\
-1 & 0 & 0 & -1 & -1 & -1 \\
\end{bmatrix}
\]
In a graph $G$ let $x$ be the number of cut-sets having arbitrary orientations. Then, we have the following definition.

**DEFINITION:** The cut-set matrix

$$Q = \begin{bmatrix} q_{ij} \end{bmatrix}_{x \times e}$$

for a graph $G$ of $e$ edges and $x$ cut-sets is defined as

$$q_{ij} = \begin{cases} 1 & \text{if edge } j \text{ in cut-set } i \text{ with } e_j, x_i \text{ "same" orientation} \\ -1 & \text{if edge } j \text{ in cut-set } i \text{ with } e_j, x_i \text{ "opposite" orientation} \\ 0 & \text{if edge } j \text{ not in cut-set } i \end{cases}$$
Consider the following graph and its seven possible cut-sets

$$Q = \begin{bmatrix}
1 & -1 & -1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & -1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & -1 & -1 & 0 \\
0 & 1 & 0 & 1 & 1 & 0 & 0 \\
1 & -1 & 0 & 0 & -1 & -1 & 0 \\
-1 & 0 & 1 & -1 & -1 & 0 & 1 \\
0 & -1 & -1 & -1 & 0 & 1 & 1
\end{bmatrix}$$
**f-Cut-set Matrix**

- **DEFINITION:** The $f$-cutset matrix $Q_f$ of a graph $G$ with respect to some tree $T$ is defined as the cut-set matrix consisting of the fundamental cut-set of $G$ only whose orientations are chosen in the same direction as that of defining branches.

- The fundamental cut-set matrix $A_f$ of a graph $G$ with respect to some tree $T$ can always be written as

$$Q_f = \begin{bmatrix}
U & Q_{f_{11}} \\
\times b & \times (e-b)
\end{bmatrix}$$

- Recall that $b = n-1$
Consider the following graph with $T = \{e_2, e_4, e_5\}$

$$Q_f = \begin{bmatrix} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{bmatrix}$$
THEOREM: If the column orderings of the circuit and incident matrices are identical then

\[ \mathbf{Q}_f \mathbf{B}_f^T = 0 \]

Also

\[ \mathbf{B}_f \mathbf{Q}^T = 0 \]

\[ \mathbf{Q}_f \mathbf{B}_f^T = 0 \]

\[ \mathbf{B} \mathbf{Q}^T = 0 \]
Consider the following graph

\[ Q = \begin{bmatrix}
1 & -1 & -1 & 0 & 0 & 0 \\
-1 & 0 & 0 & -1 & 0 & 1 \\
0 & 0 & 1 & 0 & -1 & -1 \\
0 & 1 & 0 & 1 & 1 & 0 \\
1 & -1 & 0 & 0 & -1 & -1 \\
-1 & 0 & 1 & -1 & -1 & 0 \\
0 & -1 & -1 & -1 & 0 & 1 \\
\end{bmatrix} \]

\[ B = \begin{bmatrix}
0 & 0 & 0 & 1 & -1 & 1 \\
1 & 1 & 0 & -1 & 0 & 0 \\
0 & -1 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & -1 & 1 \\
0 & -1 & 1 & 1 & 0 & 1 \\
1 & 0 & 1 & -1 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 \\
\end{bmatrix} \]
Now, let $G$ be a connected graph having $e$ edges and let

$$\mathbf{x}^T = \left[ x_1(t), x_2(t), \ldots x_e(t) \right]$$

and

$$\mathbf{y}^T = \left[ y_1(t), y_2(t), \ldots y_e(t) \right]$$

be two vectors where $x_i$ and $y_i$, $i=1,\ldots,e$, correspond to the across and through variables associated with the edge $i$ respectively.
2. POSTULATE Let $B$ be the circuit matrix of the graph $G$ having $e$ edges then we can write the following algebraic equation for the across variables of $G$ (e.g., edge voltage):

$$Bx = 0 \implies \text{KVL}$$

3. POSTULATE Let $Q$ be the cut-set matrix of the graph $G$ having $e$ edges then we can write the following algebraic equation for the through variables of $G$ (e.g., edge current):

$$Qy = 0 \implies \text{KCL}$$
Consider a graph $G$ and a tree $T$ in $G$. Let the vectors $\mathbf{v}$ and $\mathbf{i}$ partitioned as

$$\mathbf{v} = \begin{bmatrix} \mathbf{v}_{\text{link}} & \mathbf{v}_{\text{branch}} \end{bmatrix}^T; \mathbf{i} = \begin{bmatrix} \mathbf{i}_{\text{branch}} & \mathbf{i}_{\text{link}} \end{bmatrix}^T$$

Then

$$\mathbf{B}_f \mathbf{v} = \begin{bmatrix} \mathbf{U} & \mathbf{B}_{f12} \end{bmatrix} \begin{bmatrix} \mathbf{v}_{\text{link}} \\ \mathbf{v}_{\text{branch}} \end{bmatrix} = 0$$

$$\mathbf{Q}_f \mathbf{i} = \begin{bmatrix} \mathbf{U} & \mathbf{Q}_{f11} \end{bmatrix} \begin{bmatrix} \mathbf{i}_{\text{branch}} \\ \mathbf{i}_{\text{link}} \end{bmatrix} = 0$$

$$\mathbf{v}_{\text{link}} = -\mathbf{B}_{f12} \mathbf{v}_{\text{branch}}$$

$$\mathbf{i}_{\text{branch}} = -\mathbf{Q}_{f11} \mathbf{i}_{\text{link}}$$

fundamental circuit equation  fundamental cut-set equation
**Definition:** Two edges $e_i$ and $e_k$ are said to be connected in series if they have exactly one common vertex of degree two.

**Definition:** Two edges $e_i$ and $e_k$ are said to be connected in parallel if they are incident at the same pair of vertices $v_i$ and $v_k$. 
General Procedure

1. Draw a graph and then identify a tree.
2. Place all control-voltage branches for voltage-controlled dependent sources in the tree, if possible.
3. Place all control-current branches for current-controlled dependent sources in the cotree, if possible.
4. Find incidence, f-circuit, or f-cutset matrix.
5. Replace voltage, current sources with short, open circuits, respectively.
6. Formulate the matrix equation.