LE230: Numerical Technique In Electrical Engineering

Lecture 1: **Introduction to Numerical Methods**

- **What are numerical methods and why do we need them?**
- **Course outline.**
- **Number Representation**
- **Floating point number**
- **Errors in numerical analysis**
- **Taylor Theorem**

My advice

- If you don’t let a teacher know at what level you are by asking a question, or revealing your ignorance you will not learn or grow.
- You can’t pretend for long, for you will eventually be found out. Admission of ignorance is often the first step in our education.
  - Steven Covey—Seven Habits of Highly Effective People

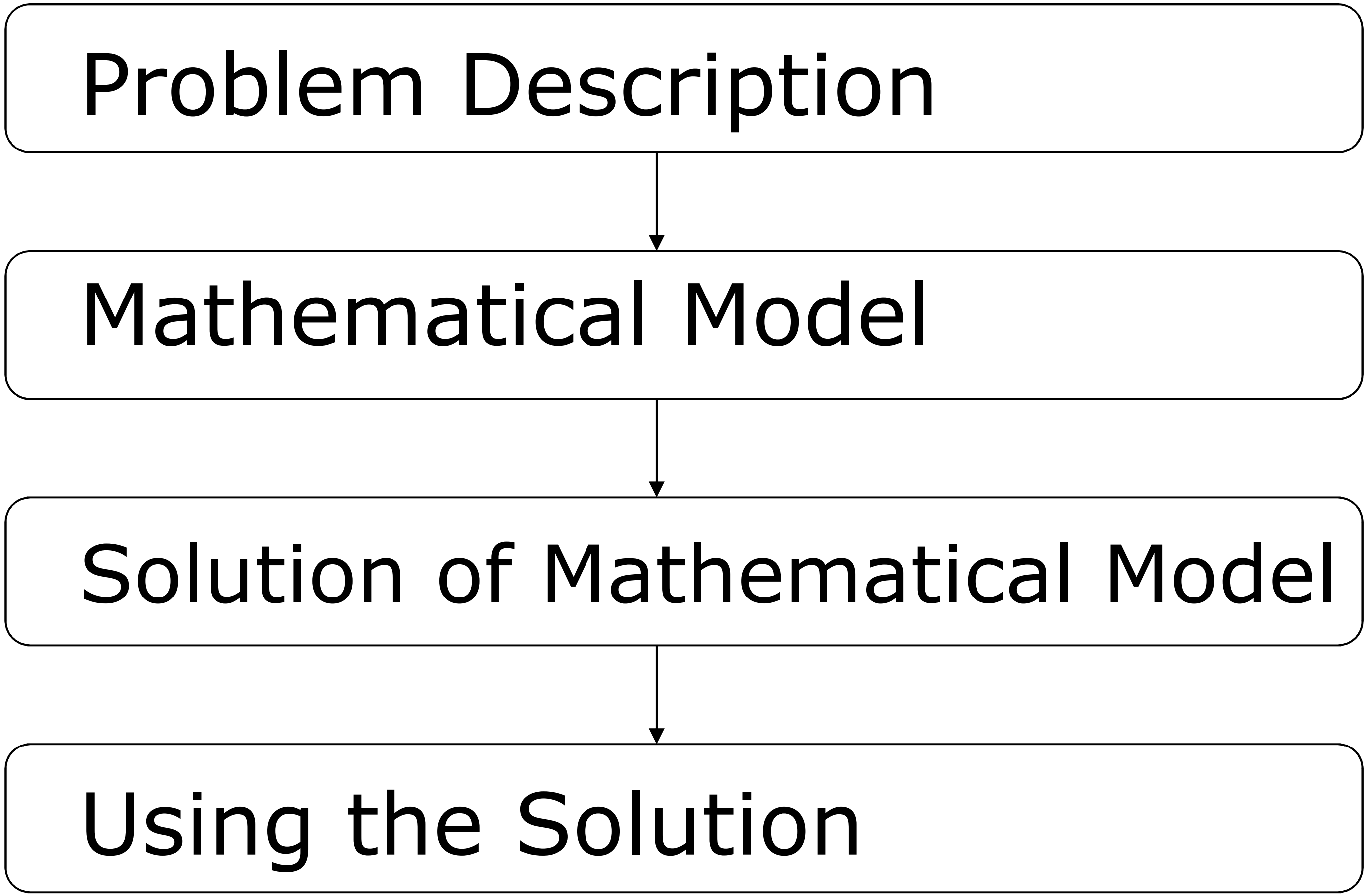
Course Objectives

- Understand numerical techniques, i.e., meaning and significance.
- Study numerical methods, i.e., Algorithms that are used to obtain numerical solutions of a mathematical problem.
- Apply numerical methods for solving engineering problems.

Expectations

- In this course, “hopefully” you’ll learn
  - Fundamentals of numerical methods
  - Basic numerical methods, e.g., solving system of equations, numerical integration, etc.
  - Implementation of numerical methods
  - Basic Programming
  - Application of numerical methods

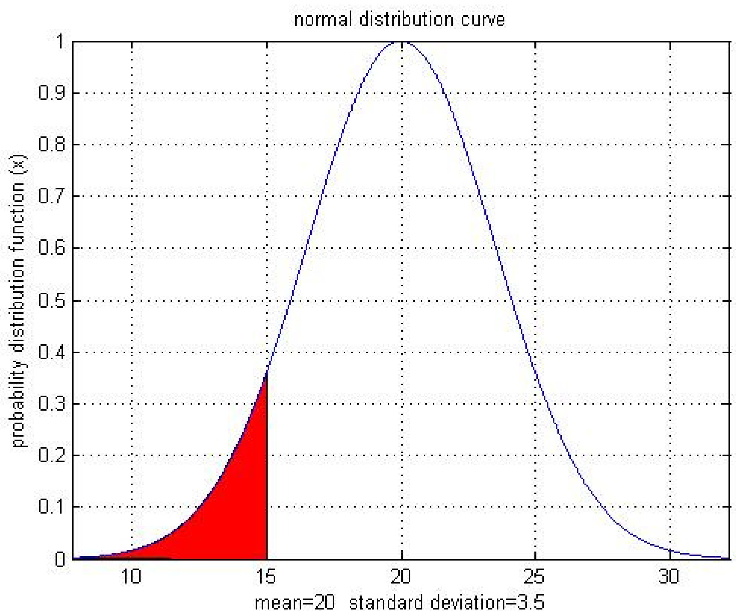
How do we solve an engineering problem?



Why use Numerical Methods?

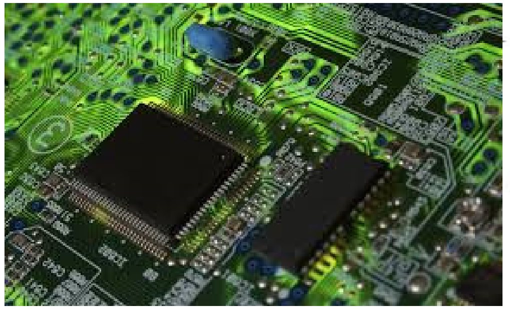
To solve problems that cannot be solved analytically (i.e., exactly) or an analytical solution is difficult to obtain or not practical.

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}} du$$



Why use Numerical Methods?

- To solve problems that are intractable!



What do we need?

Basic Needs in the Numerical Methods:

- Practical:

  Can be computed in a reasonable amount of time.
- Accurate:
  - Good approximate to the true value,
  - Information about the approximation error (Bounds, error order,…).

Outlines of the Course

- Taylor Theorem
- Number Representation
- Solution of nonlinear Equations
- Solution of linear Equations
- Regression and Interpolation
- Numerical Differentiation

- Numerical Integration
- Solution of ordinary differential equations (ODE)
- Solution of Partial differential equations (PDE)
- Eigenvalue Problem
- Graph Theory and Applications

Solution of Nonlinear Equations

- Some simple equations can be solved analytically:

$$x^2 + 4x + 3 = 0$$

$$\text{Analytic solution} \quad roots = \frac{-4 \pm \sqrt{4^2 - 4(1)(3)}}{2(1)}$$

$$x = -1 \ \ and \quad x = -3$$

- Many other equations have no analytical solution:

$$\left.\begin{array}{c} x^9 - 2x^2 + 5 = 0 \\ x = e^{-x} \end{array}\right\} \text{No analytic solution}$$

Solution of Systems of Linear Equations

$$x_1 + x_2 = 3$$

$$x_1 + 2x_2 = 5$$

We can solve it as :

$$x_1 = 3 - x_2 , \qquad 3 - x_2 + 2x_2 = 5$$

$$\Rightarrow x_2 = 2, \; x_1 = 3 - 2 = 1$$

What to do if we have

1000 equations in 1000 unknowns.

Cramer's Rule is Not Practical

Cramer's Rule can be used to solve the system:

$$x_1 = \frac{\begin{vmatrix} 3 & 1 \\ 5 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}} = 1, \qquad x_2 = \frac{\begin{vmatrix} 1 & 3 \\ 1 & 5 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}} = 2$$

But Cramer's Rule is not practical for large problems.

To solve N equations with N unknowns, we need $(N+1)(N-1)N!$ multiplications.

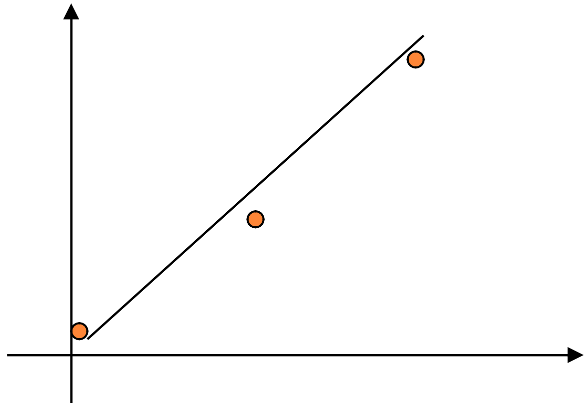
To solve a 30 by 30 system, 2.3×10^{35} multiplications are needed.

A super computer needs more than 10^{20} years to compute this.

Curve Fitting : Regression

- Given a set of data:

x	0	1	2
y	0.5	10.3	21.3

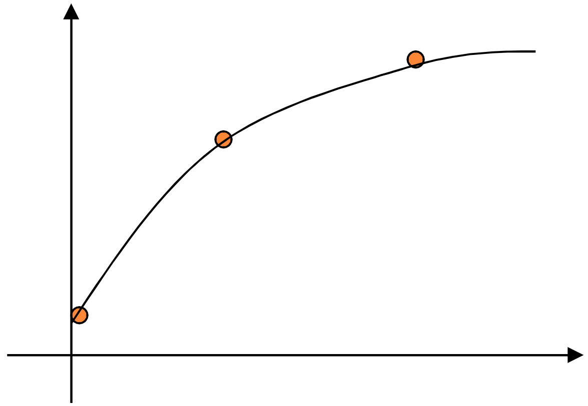


- Select a curve that best fits the data. One choice is to find the curve so that the sum of the square of the error is minimized.

Curve Fitting : Interpolation

- Given a set of data:

x_i	0	1	2
y_i	0.5	10.3	15.3



- Find a polynomial *P(x)* whose graph passes through all tabulated points.

$$y_i = P(x_i) \quad if \ x_i \text{ is in the table}$$

Integration

- Some functions can be integrated analytically:

$$\int_1^3 x dx = \frac{1}{2}x^2\Big|_1^3 = \frac{9}{2} - \frac{1}{2} = 4$$

But many functions have no analytical solutions:

$$\int_0^a e^{-x^2} dx = ?$$

Solution of Ordinary Differential Equations

A solution to the differential equation :

$$\ddot{x}(t) + 3\dot{x}(t) + 3x(t) = 0$$

$$\dot{x}(0) = 1; x(0) = 0$$

is a function *x(t)* that satisfies the equations.

* Analytical solutions are available for special cases only.

Solution of Partial Differential Equations

Partial Differential Equations are more difficult to solve than ordinary differential equations:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial t^2} + 2 = 0$$

$$u(0,t) = u(1,t) = 0,\ u(x,0) = \sin(\pi x)$$

Representing Real Numbers

- You are familiar with the decimal system:

$$312.45 = 3\times10^{2} + 1\times10^{1} + 2\times10^{0} + 4\times10^{-1} + 5\times10^{-2}$$

- Decimal System: Base = 10 , Digits (0,1,…,9)

- Standard Representations:

±	3 1 2	.	4 5
sign	integer part		fraction part

Normalized Floating Point Representation

- Normalized Floating Point Representation:

$$\pm \quad \underline{d.\, f_1\ f_2\ f_3\ f_4} \times 10^{\pm n}$$

sign mantissa (fraction) exponent

$$d \neq 0, \qquad \pm n : \text{signed exponent}$$

- Scientific Notation: Exactly one non-zero digit appears before decimal point.
- Advantage: Efficient in representing very small or very large numbers.

Binary System

- Binary System: Base = 2, Digits {0,1}

$$\pm \quad \underline{1.\ f_1 \ \ f_2 \ \ f_3 \ \ f_4} \ \times \ 2^{\pm n}$$

sign mantissa signed exponent

$$(1.101)_2 = (1 + 1 \times 2^{-1} + 0 \times 2^{-2} + 1 \times 2^{-3})_{10} = (1.625)_{10}$$

Fact

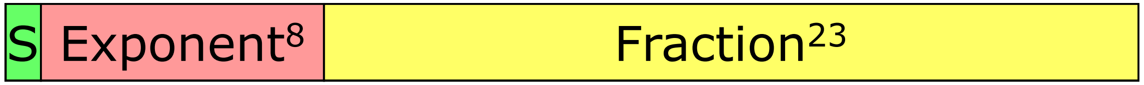
- Numbers that have a finite expansion in one numbering system may have an infinite expansion in another numbering system:

$$(1.1)_{10} = (1.0001100110\ 01100\ldots)_2$$

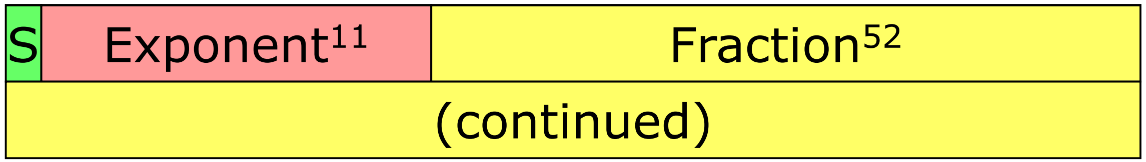
- You can never represent 1.1 exactly in binary system.

IEEE 754 Floating-Point Standard

- Single Precision (32-bit representation)
  - 1-bit Sign + 8-bit Exponent + 23-bit Fraction



- Double Precision (64-bit representation)
  - 1-bit Sign + 11-bit Exponent + 52-bit Fraction



Significant Digits

- Significant digits are those digits that can be used with confidence.
- Single-Precision: 7 Significant Digits

  $1.175494\ldots \times 10^{-38}$ to $3.402823\ldots \times 10^{38}$
- Double-Precision: 15 Significant Digits

  $2.2250738\ldots \times 10^{-308}$ to $1.7976931\ldots \times 10^{308}$

Remarks

- Numbers that can be exactly represented are called machine numbers.
- Difference between machine numbers is not uniform
- Sum of machine numbers is not necessarily a machine number

Calculator Example

- Suppose you want to compute:

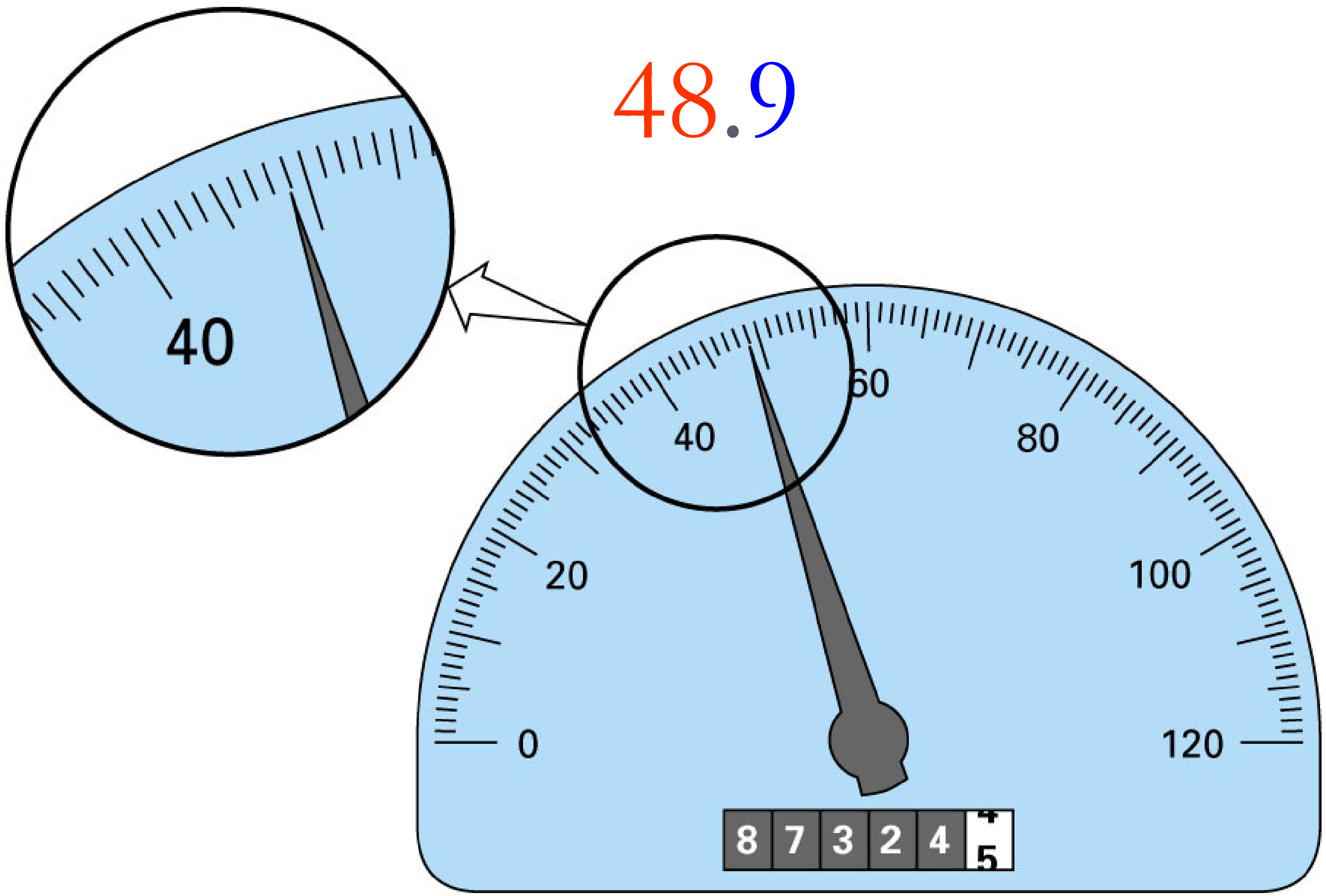
  3.578 * 2.139

  using a calculator with two-digit fractions

3.57 * 2.13 = 7.60

True answer: 7.653342

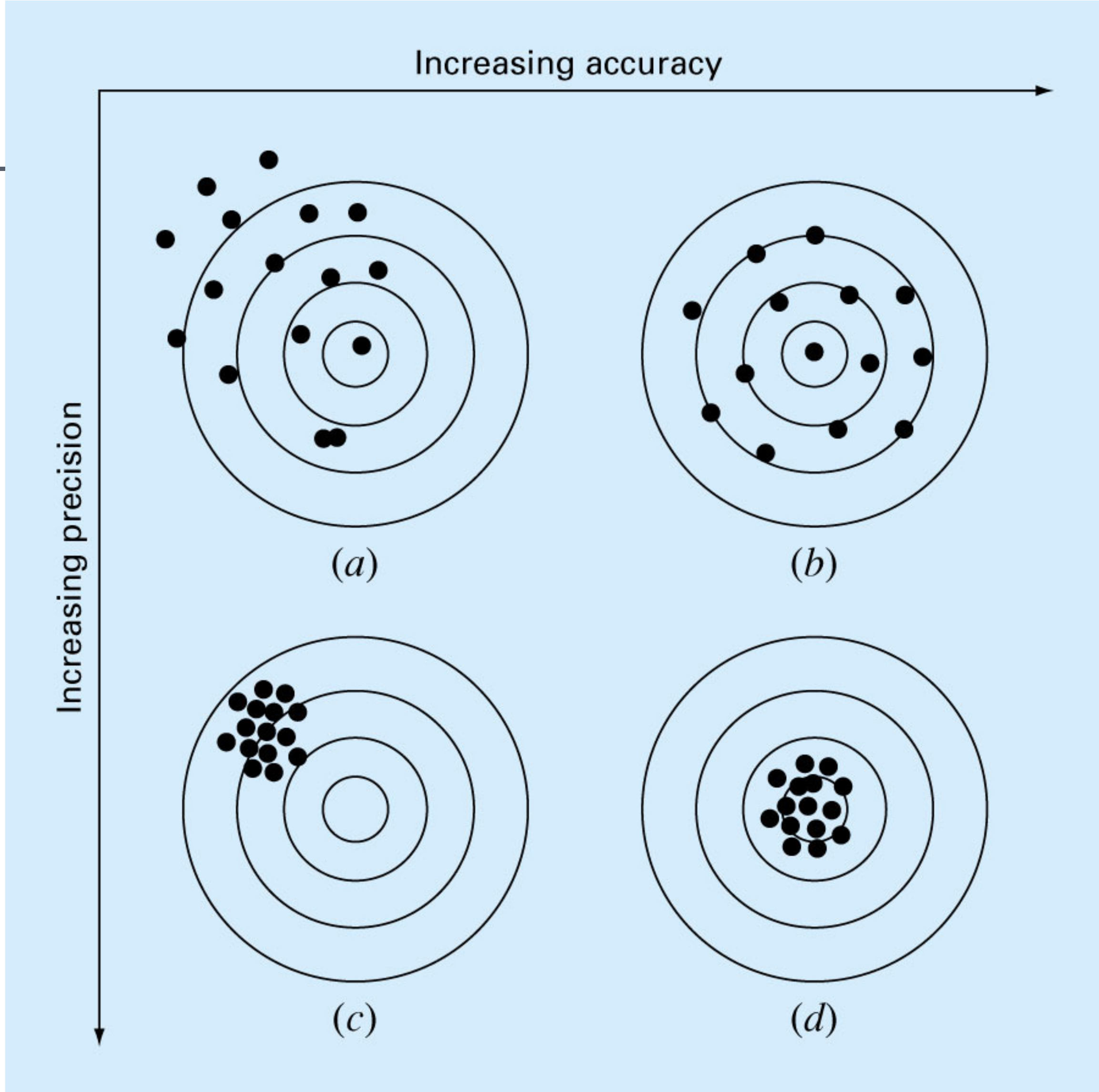
Significant Digits - Example



Accuracy and Precision

- *Accuracy* is related to the closeness to the true value.

- *Precision* is related to the closeness to other estimated values.

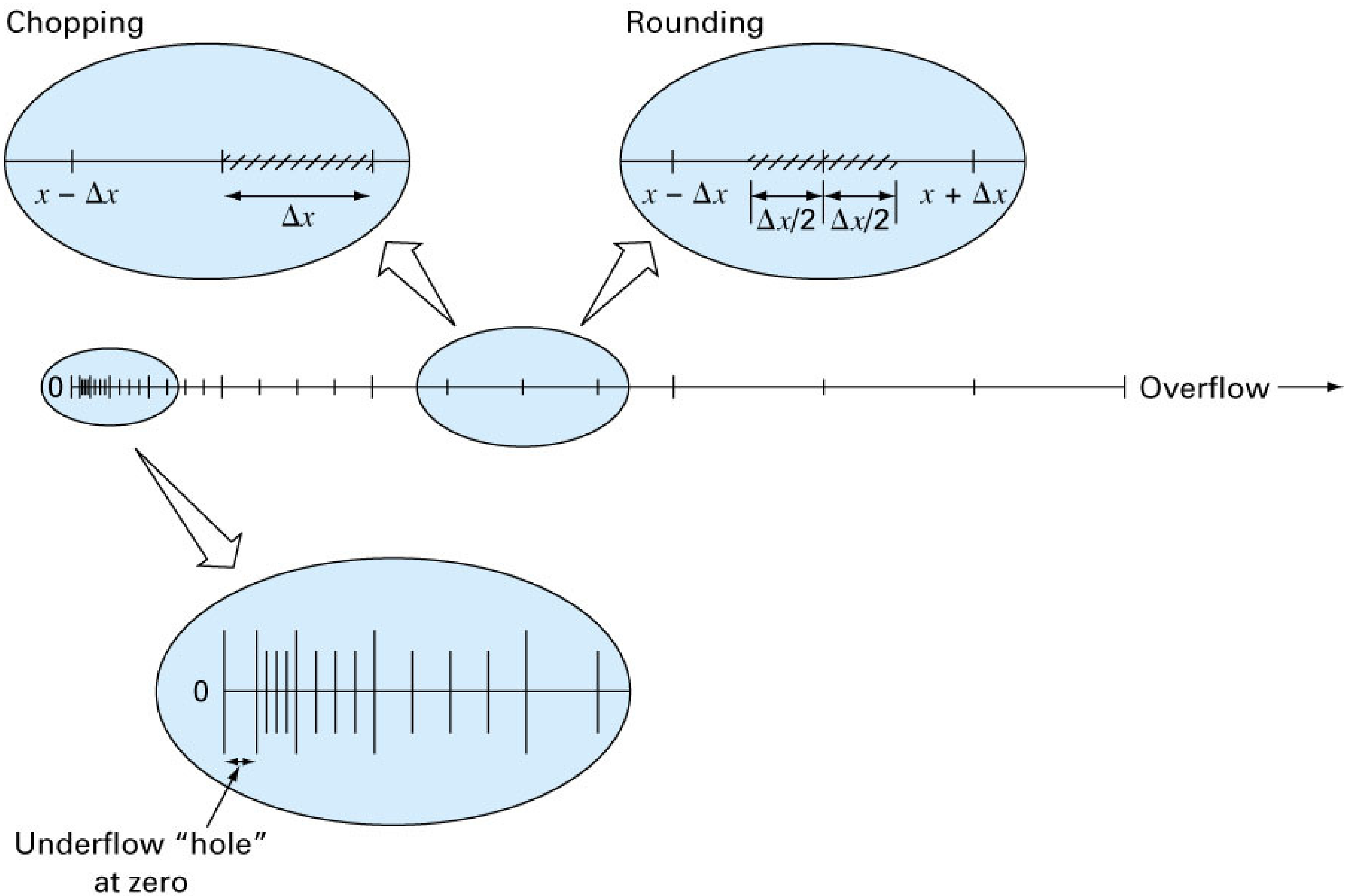

Increasing accuracy
Increasing precision
(a)
(b)
(c)
(d)

Rounding and Chopping

- Rounding: Replace the number by the nearest machine number
  - *Round-off Error*

- Chopping: Throw all extra digits.
  - *Truncation Error*

Rounding and Chopping



Error Definitions – True Error

Can be computed if the true value is known:

Absolute True Error

$$E_t = \left| \text{true value} - \text{approximation} \right|$$

Absolute Percent Relative Error

$$\varepsilon_t = \left| \frac{\text{true value} - \text{approximation}}{\text{true value}} \right| * 100$$

Error Definitions – Estimated Error

When the true value is not known:

Estimated Absolute Error

$$E_a = \left| \text{current estimate} - \text{previous estimate} \right|$$

Estimated Absolute Percent Relative Error

$$\varepsilon_a = \left| \frac{\text{current estimate} - \text{previous estimate}}{\text{current estimate}} \right| * 100$$

Notation

We say that the estimate is correct to *n* decimal digits if:

$$|\text{Error}| \le 10^{-n}$$

We say that the estimate is correct to *n* decimal digits **rounded** if:

$$|\text{Error}| \le \frac{1}{2} \times 10^{-n}$$

Loss of Significant Digits

- Subtraction of two "relatively close" numbers can lead to loss of significant digits (or significance)
- Example: Suppose 7 significant digits

$x = 0.1234567$, $y = 0.1234566$

$x - y = 0.0000001$ -> <u>1 significant digit</u>

Loss of Significant Digits Example

- Consider the following quadratic equation:

$$ax^2 + bx + c = 0; x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\text{If } b^2 >> 4ac, b \approx \sqrt{b^2 - 4ac}$$

- Example: *a*=1, *b*=1111.11, *c*=1.2121 and assume 7 significant digits:

$$b^2 = 1234565 >> 4ac = 4.8484, b^2 - 4ac = 1234560$$

$$\sqrt{b^2 - 4ac} = 1111.108, x_1 = -0.001000 \neq -0.001091$$

- Can use $x_1 = -2c/(b + \sqrt{b^2 - 4ac}) = -0.001091$

Taylor Series

The Taylor series expansion of $f(x)$ about a:

$$f(a) + f^{'}(a)(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \ldots$$

or

$$Taylor\ \ Series = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(a)\ (x-a)^k$$

If the series converge, we can write :

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(a)\ (x-a)^k$$

Maclaurin Series

- Maclaurin series is a special case of Taylor series with the center of expansion *a* = 0.

$$\text{The Maclaurin series expansion of } f(x):$$

$$f(0) + f'(0)\,x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \ldots$$

$$\text{If the series converge, we can write:}$$

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(0)\; x^k$$

Maclaurin Series – Example 1

Obtain Maclaurin series expansion of $f(x) = e^x$

$$f(x) = e^x \qquad f(0) = 1$$

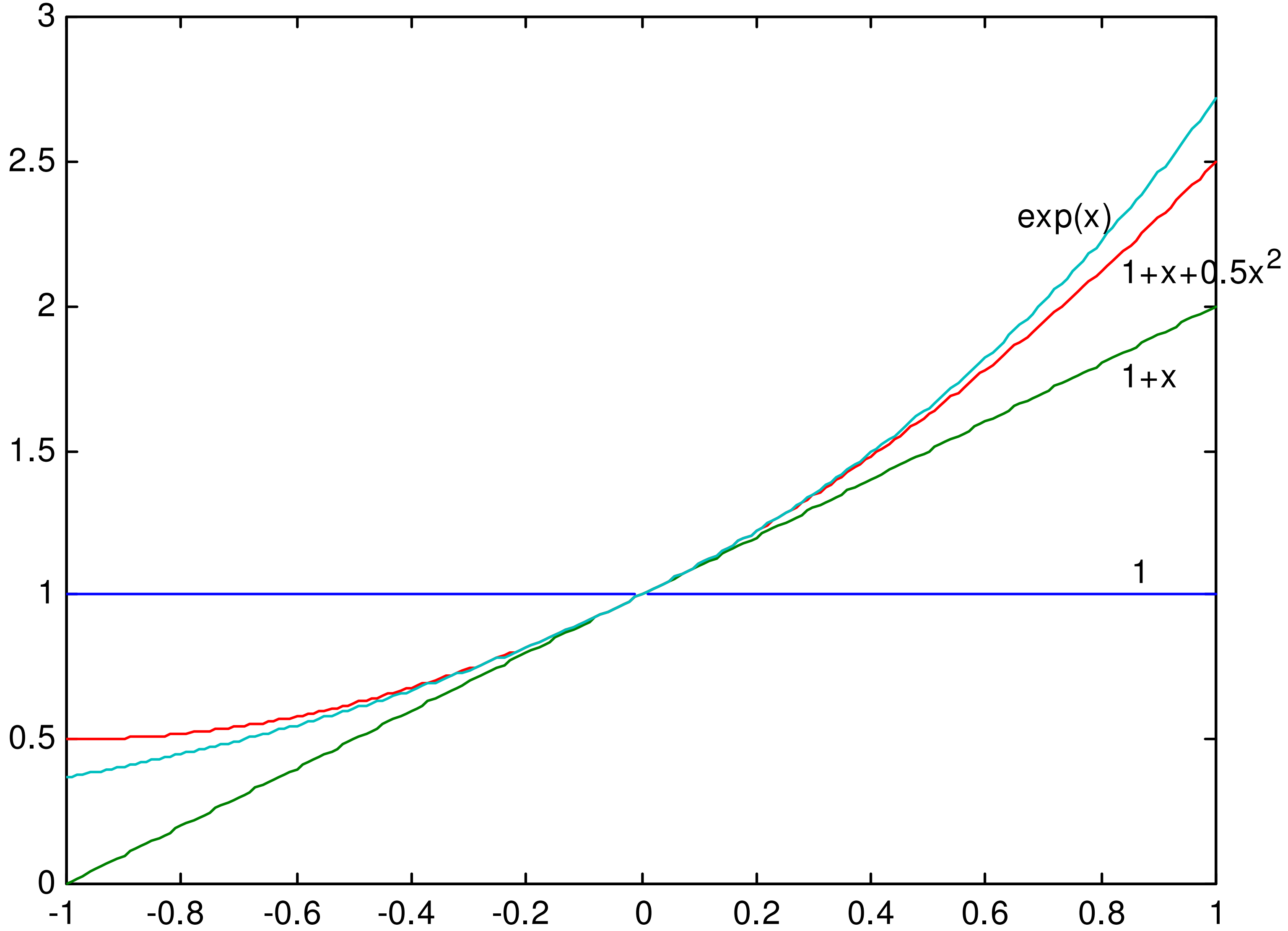
$$f'(x) = e^x \qquad f'(0) = 1$$

$$f^{(2)}(x) = e^x \qquad f^{(2)}(0) = 1$$

$$f^{(k)}(x) = e^x \qquad f^{(k)}(0) = 1 \quad for\, k \geq 1$$

$$e^x = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(0)\; x^k = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

The series converges for $|\mathrm{x}| < \infty$.



3
2.5
2
1.5
1
0.5
0
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
exp(x)
1+x+0.5x2
1+x
1

Maclaurin Series – Example 2

Obtain Maclaurin series expansion of $f(x) = \sin(x)$:

$$f(x) = \sin(x) \qquad f(0) = 0$$

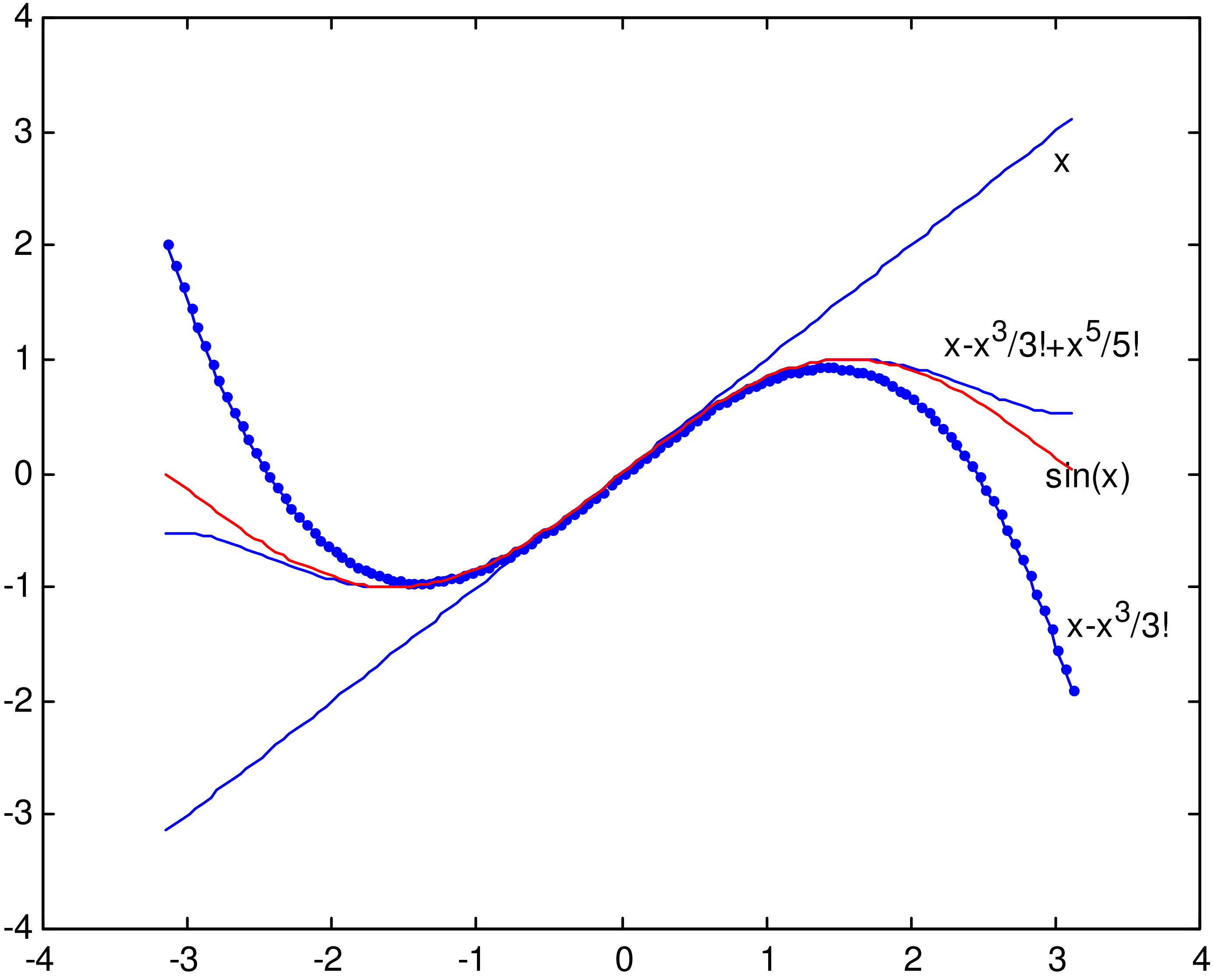
$$f'(x) = \cos(x) \qquad f'(0) = 1$$

$$f^{(2)}(x) = -\sin(x) \qquad f^{(2)}(0) = 0$$

$$f^{(3)}(x) = -\cos(x) \qquad f^{(3)}(0) = -1$$

$$\sin(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots.$$

The series converges for $|x| < \infty$.


4
3
2
1
0
-1
-2
-3
-4
-4
-3
-2
-1
0
1
2
3
4
x
x-x^3/3!+x^5/5!
sin(x)
x-x^3/3!

Convergence of Taylor Series

- The Taylor series converges fast (few terms are needed) when ***x*** is near the point of expansion. If **|*x-a*|** is large, then more terms are needed to get a good approximation.

Taylor's Theorem

If a function $f(x)$ possesses derivatives of orders $1, 2, \ldots, (n+1)$ on an interval containing a and x then the value of $f(x)$ is given by :

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + R_n$$

(n+1) terms Truncated Taylor Series

Remainder

where :

$$R_n = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1} \quad and \ \xi \text{ is between } a \text{ and } x.$$

Taylor's Theorem

We can apply Taylor's theorem for :

$$f(x)=\frac{1}{1-x} \quad \text{with the point of expansion } a=0 \text{ if } |x|<1.$$

If $x=1$, then the function and its derivatives are not defined.

$\Rightarrow$ Taylor Theorem is not applicable.

Error Term

To get an idea about the approximation error, we can derive an upper bound on:

$$R_n = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$$

for all *values of* ξ between a and x.

Error Term - Example

How large is the error if we replaced $f(x) = e^x$ by the first 4 terms $(n = 3)$ of its Taylor series expansion at $a = 0$ $\quad when \ x = 0.2$?

$$f^{(n)}(x) = e^x \qquad f^{(n)}(\xi) \leq e^{0.2} \quad for\, n \geq 1$$

$$R_n = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$$

$$\left|R_n\right| \leq \frac{e^{0.2}}{(n+1)!}(0.2)^{n+1} \Rightarrow \left|R_3\right| \leq 8.14268E-05$$

Alternative form of Taylor's Theorem

Let $f(x)$ have derivatives of orders $1, 2, ..., (n+1)$ on an interval containing x and $x+h$ then :

$$f(x+h) = \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!} h^k + R_n \qquad (h = \text{step size})$$

$$R_n = \frac{f^{(n+1)}(\xi)}{(n+1)!} h^{n+1} \quad \textit{where } \xi \text{ is between } x \text{ and } x+h$$

Taylor's Theorem – Alternative forms

$$f(x)=\sum_{k=0}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k+\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$$

where ξ is between a and x.

$$a\to x,\quad x\to x+h$$

$$f(x+h)=\sum_{k=0}^{n}\frac{f^{(k)}(x)}{k!}h^k+\frac{f^{(n+1)}(\xi)}{(n+1)!}h^{n+1}$$

where ξ is between x and $x+h$.

Mean Value Theorem

If $f(x)$ is a continuous function on a closed interval $[a,b]$ and its derivative is defined on the open interval (a,b) then there exists $\xi \in (a,b)$

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}$$

Proof : Use Taylor's Theorem for $n = 0,\ x = a,\ x + h = b$

$$f(b) = f(a) + f'(\xi)(b - a)$$

Alternating Series Theorem

Consider the alternating series:

$$S = a_1 - a_2 + a_3 - a_4 + \cdots$$

If $\left\{ \begin{array}{l} a_1 \geq a_2 \geq a_3 \geq a_4 \geq \cdots \\ \quad \textit{and} \\ \lim_{n \to \infty} a_n = 0 \end{array} \right.$ then $\left\{ \begin{array}{c} \text{The series converges} \\ \textit{and} \\ |S - S_n| \leq a_{n+1} \end{array} \right.$

S_n : Partial sum (sum of the first n terms)

a_{n+1} : First omitted term

Alternating Series – Example

sin(1) can be computed using: $sin(1) = 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \cdots$

This is a convergent alternating series since:

$a_1 \geq a_2 \geq a_3 \geq a_4 \geq \cdots \; and \; \lim_{n \to \infty} a_n = 0$

Then:

$$\left| sin(1) - \left(1 - \frac{1}{3!}\right) \right| \leq \frac{1}{5!}$$

$$\left| sin(1) - \left(1 - \frac{1}{3!} + \frac{1}{5!}\right) \right| \leq \frac{1}{7!}$$

Example 3 – Taylor Series

Obtain Taylor series expansion of $f(x) = e^{2x+1}$, $a = 0.5$

$$f(x) = e^{2x+1} \qquad f(0.5) = e^2$$

$$f'(x) = 2e^{2x+1} \qquad f'(0.5) = 2e^2$$

$$f^{(2)}(x) = 4e^{2x+1} \qquad f^{(2)}(0.5) = 4e^2$$

$$f^{(k)}(x) = 2^k \ e^{2x+1} \qquad f^{(k)}(0.5) = 2^k e^2$$

$$e^{2x+1} = \sum_{k=0}^{\infty} \frac{f^{(k)}(0.5)}{k!} (x - 0.5)^k$$

$$= e^2 + 2e^2(x - 0.5) + 4e^2 \frac{(x - 0.5)^2}{2!} + \ldots + 2^k e^2 \frac{(x - 0.5)^k}{k!} + \ldots$$

Example 3 – Error Term

$$f^{(k)}(x) = 2^k \ e^{2x+1}$$

$$Error = \frac{f^{(n+1)}(\xi)}{(n+1)!}\ (x-0.5)^{n+1}$$

$$\left|Error\right| = \left| 2^{n+1} e^{2\xi+1} \quad \frac{(1-0.5)^{n+1}}{(n+1)!} \right|$$

$$\left|Error\right| \le 2^{n+1} \frac{(0.5)^{n+1}}{(n+1)!} \max_{\xi \in [0.5,1]} \left| e^{2\xi+1} \right|$$

$$\left|Error\right| \le \frac{e^3}{(n+1)!}$$