Network Theorems

1 Superposition Theorem

The basic principle of superposition states that, if the effect produced in a system is directly proportional to the cause, then the overall effect produced in the system, due to a number of causes acting jointly, can be determined by superposing (adding) the effects of each source acting separately. This principle is only applicable to “linear” networks and system.

Consider the circuits below. In Fig. 1, clearly the voltage across the resistor and the current are given by

\[ V_R = V_1 - V_2; \quad I = \frac{V_1}{R} - \frac{V_2}{R}. \]

Now, applying the superposition theorem yields

\[ I = I_1 + I_2 = \frac{V_1}{R} - \frac{V_2}{R}, \]

i.e., the sum of the currents due to two sources.

**Question** Is the superposition theorem applicable to the power as well?

**Example 1** Verify the superposition theorem.

![Fig. 3: Example 1 problem](image1)

**Example 2** Verify the superposition theorem.

![Fig. 4: Example 2 problem](image2)

2 Reciprocity Theorem

Consider two loops A and B of a network N where an ideal voltage source V in loop A produces a current I in loop B, then the network is said to be reciprocal if an identical source in loop B produces the same current I in loop A. In short, a linear network is said to be reciprocal if it remains invariant due to the interchange of position of cause (source) and effect (linear elements) in the network.
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Example 3 Verify the reciprocity theorem.

Fig. 5: Example 3 problem

3 Thevenin’s Theorem

This theorem states that a linear circuit containing one or more sources and other linear elements can be represented by a voltage source $V_{TH}$ in series with an impedance $Z_{TH}$, $V_{TH}$ is the open-circuit voltage between the terminals of the network and $Z_{TH}$ is the impedance measured between the terminals of the network with all sources removed (but not their impedances). This is also called the voltage source equivalent circuit.

Fig. 6: General network

Example 4 Find the Thevenin’s equivalent circuit.

Fig. 8: Example 4 problem

Example 5 Find the Thevenin’s equivalent circuit.

Fig. 9: Example 5 problem
4 Norton’s Theorem
Norton’s theorem says that the linear network consisting of one or more independent sources and linear elements can be represented by a current source $I_{SC}$ and an equivalent impedance $Z_{TH}$ in parallel with the current source. $I_{SC}$ is the short-circuit current between the terminals of the network and $Z_{TH}$ is the impedance measured between the terminals with all sources removed (but not their impedances). This is also called the current source equivalent circuit.
Example Repeat examples 4, 5 using Norton’s equivalent circuits.

5 Millman’s Theorem
Let $V_i$ ($i=1,2,...,n$) be the open-circuit voltages of $n$ voltage sources having internal impedances $Z_i$ in series, respectively, as shown in Fig. 10. Suppose these sources are connected in parallel, then they may be replaced by a single ideal voltage source $V$ in series with an impedance $Z$, where

$$V = \frac{\sum_{i=1}^{n} V_i Y_i}{\sum_{i=1}^{n} Y_i}; Z = \frac{1}{\sum_{i=1}^{n} Y_i}.$$  

**Proof**

**Example 6** Find the current $I$ in Fig. 11.

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**Fig. 10: Millman’s Theorem**

**Fig. 11: Example 6 problem**
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6 Maximum Power Transfer Theorem
Maximum power will be delivered to a network, to an impedance $Z_L$, if the impedance of $Z_L$ is the complex conjugate of the impedance $Z$ of the network, measured looking back into the terminals of the network.

**Derivation**

Example 7 A circuit model of a transistor driven by a current source $i(t)$ is shown in Fig. 12, where $R_S$ is the source internal impedance and $h_v, h_i, h_f$ and $1/h_0$ are transistor parameters. Find Thevenin’s and Norton’s equivalent circuits and derive the condition of maximum power transfer.

![Fig. 12: Example 7 problem](image)

7 Substitution Theorem
Sometimes, it is convenient to replace an impedance branch by another branch with different circuit components, without disturbing the voltage-current relationship in the network. The condition under which, branch replacement is possible, is given by the substitution theorem. It states that any branch in a network may be substituted by a different branch without disturbing the voltages and currents in the entire network, provided the new branch has the same set of terminal voltage and current as the original branch.

The substitution theorem is a general theorem and is applicable for any arbitrary network. It is very useful in circuit analysis of networks having one non-linear element. Also, it is often used to replace the effect of mutual inductance.

Example 8 Find the substitutions for xy branch.

![Fig. 13: Example 8 problem](image)
**8 Compensation Theorem**

In some problems, we are interested in finding the corresponding changes in various voltages and currents of a network subjected to a change in one of its branches. The compensation theorem provides us a convenient method for determining such effects.

In a linear network $N$, if the current in a branch is $I$ and the impedance $Z$ of the branch is increased by $\Delta Z$, then the increment of voltage and current in each branch of the network is that voltage or current that would be produced by an opposing voltage source of value $V_c (=I\Delta Z)$ introduced into the altered branch after the modification. The compensation theorem is based on the superposition principle, and the network is required to be linear.

Consider the network $N$ in Fig. 14, having branch impedance $Z$, then

$$I = \frac{V_{oc}}{Z + Z_{TH}}.$$  

Let $\delta Z$ be the change in $Z$. Then $I'$ (the new current) can be written as

$$I' = \frac{V_{oc}}{Z + \delta Z + Z_{TH}},$$

as shown in Fig. 15. It follows that

$$\delta I = I' - I = \frac{V_{oc}}{Z + \delta Z + Z_{TH}} - \frac{V_{oc}}{Z + Z_{TH}}$$

$$= -\frac{V_{oc}}{Z + \delta Z + Z_{TH}} \cdot \frac{\delta Z}{Z + \delta Z + Z_{TH}} = -\frac{I\delta Z}{Z + \delta Z + Z_{TH}}$$

where $V_c = I\delta Z$, which is shown in Fig. 16.

**Example 9** Verify the compensation theorem when $R$ is changed from 4 to 2 Ω.

![Fig. 14: Original Thevenin’s equivalent circuit](image)

![Fig. 15: Load is changed by $\delta Z$.](image)

![Fig. 16: Equivalent circuit by the Compensation theorem](image)

![Fig. 17: Example 9 problem](image)
9 Tellegen’s Theorem

Tellegen’s theorem is based on two Kirchhoff’s laws and is applicable for any lumped network having elements which are linear or non-linear, active or passive, time-varying or time-invariant. It is completely independent of the nature of elements and is only concerned with the graph of the network. Consider an arbitrary lumped network whose graph $G$ has $b$ branches and $n$ nodes. Suppose, to each branch of the graph, we assign arbitrarily a branch voltage $v_k$ and a branch current $i_k$ for $k = 1, 2, \ldots, b$ and suppose that they are measured with respect to arbitrarily chosen associated reference directions.

If the branch voltages $v_1, v_2, \ldots, v_b$ satisfy all the conditions imposed by KVL and if the branch currents $i_1, i_2, \ldots, i_b$ satisfy all the constraints imposed by KCL, then

$$\sum_{k=1}^{b} v_k(t) i_k(t') = 0, \forall t, t'.$$

It is noted that in a linear time-invariant network composed of energy sources and passive elements under steady-state sine-wave excitation, the conservation of power is depicted by Tellegen’s theorem with $t = t'$.

Proof Consider a network $N$ consisting of $b$ branches and $n$ nodes, let $A = [a_{kj}]$ be the incidence matrix, whose elements are given by

$$a_{kj} = \begin{cases} +1, & \text{if branch } i_j \text{ leaves node } k \\ -1, & \text{if branch } i_j \text{ enters node } k \\ 0, & \text{otherwise} \end{cases}$$

Then KCL can be written as

$$A \mathbf{i} = \mathbf{0},$$

where $\mathbf{i} = [i_1, i_2, \ldots, i_b]^T$ denotes the branch-current vector. Note that $A$ is an $n \times b$ matrix. Also, the branch voltages $\mathbf{v} = [v_1, v_2, \ldots, v_b]^T$ are related to the node voltages $\mathbf{v}_n = [v_{n1}, v_{n2}, \ldots, v_{nm}]^T$ by

$$\mathbf{v} = A^T \mathbf{v}_n.$$

Therefore,

$$\sum_{k=1}^{b} v_k(t) i_k(t') = \mathbf{v}^T \mathbf{i} = (A^T \mathbf{v}_n)^T \mathbf{i} = \mathbf{v}_n^T A \mathbf{i} = 0. \ (QED)$$

Furthermore, consider another network $N'$, which has the same topological configuration, the same references for branch currents and voltages, and the same numbering for the branches as the network $N$. Consequently, both networks have the same incidence matrix $A$, and it follows that

$$A \mathbf{i}' = \mathbf{0}; \ \mathbf{v}' = A^T \mathbf{v}_n',$$

where $\mathbf{i}'$, $\mathbf{v}'$, $\mathbf{v}_n'$ denote branch current vector, branch voltage vector and node voltage vector, of the network $N'$, respectively. Therefore,

$$\sum_{k=1}^{b} v_k(t) i_k(t') = \mathbf{v}^T \mathbf{i}' = (A^T \mathbf{v}_n')^T \mathbf{i}' = \mathbf{v}_n'^T A \mathbf{i}' = 0,$$

which implies that Tellegen’s theorem is applicable to two networks with the same topological configuration as well. Furthermore, it can be easily shown that

$$\mathbf{v}^T \mathbf{i} = \mathbf{v}_n'^T \mathbf{i}' = \mathbf{v}'^T \mathbf{i}' = 0.$$

Example 10 Find all branch currents and voltages for both networks $N_1$, $N_2$ in Fig. 18, 19. Then verify Tellegen’s theorem.

Fig. 18: Network $N_1$ in example 10 problem

Fig. 19: Network $N_2$ in example 10 problem
Example 11 Verify Tellegen’s theorem for networks \( N_1, N_2 \) in Fig. 20, 21. Assume steady-state conditions.

\[
\begin{align*}
\sum_{k=1}^{N} v_{ak} i_{k} &= v_{a0} i_{a0} + v_{a1} i_{a1} + \sum_{k=2}^{N} v_{bk} i_{ak} = \sum_{k=2}^{N} v_{bk} i_{ak} + \sum_{k=2}^{N} i_{ak} v_{bk} z_k(s) \\
\sum_{k=1}^{N} i_{k} v_{ak} &= i_{a0} v_{a0} + i_{a1} v_{a1} + \sum_{k=2}^{N} i_{bk} v_{ak} = \sum_{k=2}^{N} i_{bk} v_{ak} + \sum_{k=2}^{N} i_{ak} v_{bk} z_k(s) = v_{a0} i_{b0} + v_{a1} i_{b1}
\end{align*}
\]

Therefore we have a reciprocity of the reverse open-circuit voltage transfer equaling the forward short-circuit current transfer: \( \frac{v_{ab}}{i_{b0}} = -\frac{i_{b1}}{i_{b0}} \).

Example 12 Consider two networks with the same topology and, inside their respective two-port boxes, the same set of elements—passive complex impedances \( z_n(s) \). The outside elements differ—an open circuit at port 0 and a source at port 1 in one case, and a source at port 0 and a short circuit at port 1 in the other case.