Mason's Gain Formula

To reduce a block diagram, one may use the *Mason's gain formula* (also called Mason's rule). Mason first derived the idea using what he called a signal-flow graph, which is a different graphical representation of a block diagram. A signal-flow graph is drawn with paths (lines) and nodes. The transfer functions in a block diagram become the paths and the variables in between the blocks become the nodes. The input and output variables of a block diagram are designated the source and sink nodes. In this brief introduction, we shall skip drawing the signal-flow graphs because we can explain and apply Mason's formula to a block diagram just as well.

This is the simple idea behind the formula. We know that a block diagram is a graphical representation of algebraic relations. If we write out the equations, we should be able to solve them with, for example, the Cramer's rule. If we analyze and compare carefully the determinant terms resulting from the use of Cramer's rule with a block diagram, we may make some meaningful associations between the algebra and the diagram, and this is what Mason did. So we now state the Mason's gain formula without proof.[1] The rule states that the transfer function between the input and output variables of a block diagram is

$$G(s) = \frac{1}{\Delta} \sum_{i=1}^{f} F_i \Delta_i \,, \tag{1}$$

where Δ is the determinant of the system, F_i is the gain of the i-th forward path, and Δ_i is the determinant of the i-th forward path. The summation is over all f forward paths; we are superimposing all the terms in a linear system. Moreover, the determinant Δ is the characteristic polynomial of the system.

We now need to define some more terms and show how each of these quantities can be calculated:

System determinant	Δ = 1 – (sum of all individual loop gains) + (sum of the products of the gains of all possible *two* loops that do *not* touch each other) – (sum of the products of the gains of all possible *three* loops that do *not* touch each other) + … and so forth with sums of higher number of non-touching loop gains
Forward path gain	F_i = product of all the transfer functions along the i-th forward path
Forward path determinant	Δ_i = value of Δ for the part of the block diagram that does *not* touch the i-th forward path (Δ_i = 1 if there are no non-touching loops to the i-th path.)
Forward path	A path that goes from the input to the output, and in a way that no variables (nodes) are encountered more than once.
Loop path	A path that leads from one variable and back to the same variable.
Path gain	The forward path gain is the product of all the transfer functions along the path. Similarly, the loop path gain is the product of all the transfer functions that form the loop.
Non-touching loop	Two loops are not touching if they do not share a common variable.

[1] Hardly any introductory text provides the proof, but the text by Phillips and Harbor (1996) has a nice example to illustrate the association of the determinants with Cramer's rule.

To see how to apply the rule, we need to revisit our examples in the text. Before we do that, be forewarned that *it is extremely easy to make an error* applying the Mason's formula; we can easily overlook and omit one of the terms. We need to apply the rule with extreme care.

To apply Mason's formula, we first identify the variables in the block diagram. They are denoted with encircled numbers in the block diagrams of the following examples. Generally, we have a new variable when information is changed, either after a transfer function or after a summing point. We also label the input and output variables. It is not a strict rule, but we usually assign the numbers along the most obvious forward path first.

Example 1. Find the closed loop transfer function of a simple feedback loop (Fig. E.1).

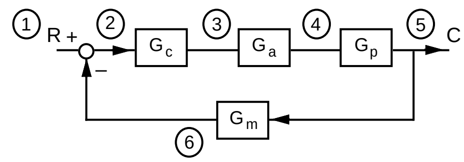

Figure E.1

This problem is essentially the block diagram in Fig. 2.11 in the text with the servo transfer function derived in Section 5.2.1. It is a good habit to make a table of the paths and loops in order to avoid errors. For this problem, there are no non-touching loops. We have only one forward path, and one loop that begins and ends after the summing point at variable number 2. The loop gain is negative because the minus sign is essentially a gain of –1.

Forward path	Path gain	Determinant
12345	$F_1 = G_cG_aG_p$	$\Delta_1 = 1$

Loop	Loop gain
234562	$G_cG_aG_pG_m \times -1$

So we have

$$\Delta = 1 - (-G_cG_aG_pG_m)\ ,$$

and since there is only one forward path, we arrive at

$$G(s) = \frac{G_c\,G_a G_p}{1 + G_c\,G_a\,G_p G_m}\ .$$

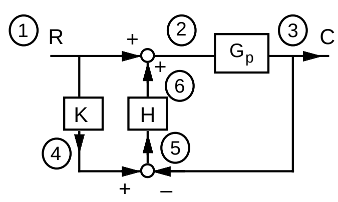

Figure E.2

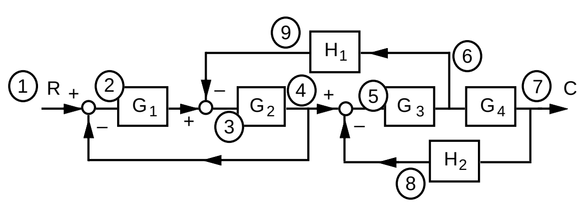

Figure E.3

Example 2. Repeat Example 2.14 in the text.

Figure E2.14 is duplicated in Fig. E.2 with the locations of the variables added. There are two forward paths and one loop path, all touching each other. There are no non-touching parts. So we have:

Forward path	Path gain	Determinant
123	$F_1 = G_p$	$\Delta_1 = 1$
145623	$F_2 = KHG_p$	$\Delta_2 = 1$

Loop	Loop gain
23562	$G_p \times -1 \times H$

The system determinant is

$$\Delta = 1 - (-G_pH) \ ,$$

and for the two forward paths,

$$\sum_{i=1}^{2} F_i \Delta_i = G_p + KHG_p \ .$$

Finally, with Eq. (1),

$$G(s) = \frac{G_p(1 + KH)}{1 + G_pH} \ .$$

Example 3. Repeat Example 2.15 in the text.

Figure E2.15a is duplicated in Fig. E.3 with the locations of the variables added. (Strictly, we should assign a variable label immediately after the block G_1, but we cheat and skip that because omitting that label will not affect our results here.) There is one forward path and three loop paths. Two of the loop paths do not touch each other, but all three loop paths touch the forward path.

Forward path	Path gain	Determinant
1234567	$F_1 = G_1G_2G_3G_4$	$\Delta_1 = 1$

Loop	Loop gain
345693	$G_2G_3H_1 \times -1$
56785*	$G_3G_4H_2 \times -1$
2342*	$G_1G_2 \times -1$

* These two loops do not touch each other

Because two of the loop paths do not touch each other, the system determinant has an extra product term of these two non-touching loops:

$$\Delta = 1 + (G_2G_3H_1 + G_3G_4H_2 + G_1G_2) + (G_3G_4H_2 \times G_1G_2) \ .$$

The forward path touches all three loops, and $\Delta_1 = 1$. Hence, the transfer function of this system is

$$G(s) = \frac{G_1 G_2 G_3 G_4}{1 + G_2 G_3 H_1 + G_3 G_4 H_2 + G_1 G_2 + G_1 G_2 G_3 G_4 H_2}.$$

If we factor out $(1 + G_1G_2)$ in the denominator, we can arrive at exactly the same form as presented in Example 2.15.

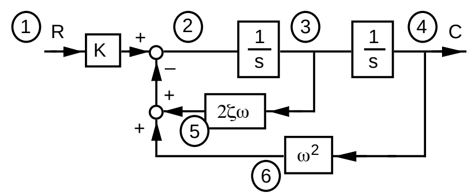

Figure E.4

Example 4. Repeat Example 2.16 in the text.

Figure E2.15(a) is duplicated in Fig. E.4 with the locations of the variables added. There is one forward path and two loop paths, all touching each other. There are no non-touching parts. So we have:

Forward path	Path gain	Determinant
1234	$F_1 = K/s^2$	$\Delta_1 = 1$

Loop	Loop gain
2352	$2\zeta\omega/s \times -1$
23462	$\omega^2/s^2 \times -1$

The system determinant is

$$\Delta = 1 + 2\zeta\omega/s + \omega^2/s^2$$

With the forward path, the transfer function via Eq. (1) is

$$G(s) = \frac{K/s^2}{1 + 2\zeta\omega/s + \omega^2/s^2} = \frac{K}{s^2 + 2\zeta\omega s + \omega^2}$$

Suggested exercises:

- Try derive the load transfer functions in examples 1 and 2 here.
- Try applying the formula to the block diagram homework problems.